

Closing Thurs: 14.1  
 Closing Fri: 14.3(1)  
 Closing *next* Tues: 14.3(2), 14.4  
 Closing *next* Thur: 14.7

Contour Map (Elevation Map) of  
 Mt. St. Helens from 1979:

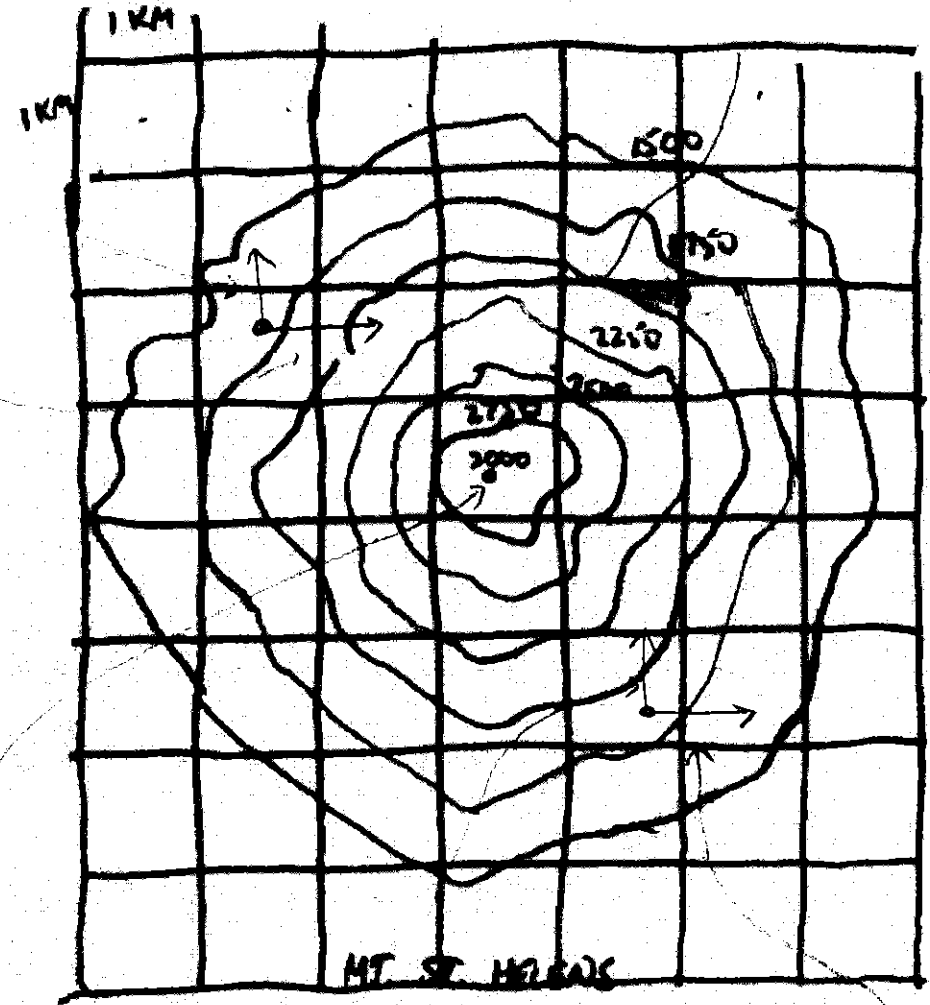
### 14.1/14.3 Visualizing Surfaces and Partial Derivatives

The basic tool for visualizing surfaces is **traces**. When  $z = f(x, y)$ , we look at traces for fixed  $z$ -values (heights) first. We call these traces **level curves**.

A collection of level curves is called a **contour map (or elevation map)**.

LOCAL MAX  
 (SLOPE ZERO IN ALL-DIRECTIONS HERE)

SLOPE NEG. IN y-DIR  
 SLOPE POSITIVE IN x-DIR



SLOPE POSITIVE IN y-DIRECTION HERE

SLOPE NEGATIVE IN x-DIRECTION HERE

Example: Draw a contour map for

$$z = f(x, y) = y - x$$

$$z=0$$

$$0 = y - x \Rightarrow y = x$$

$$z=1$$

$$1 = y - x \Rightarrow y = x + 1$$

$$z=2$$

$$2 = y - x \Rightarrow y = x + 2$$

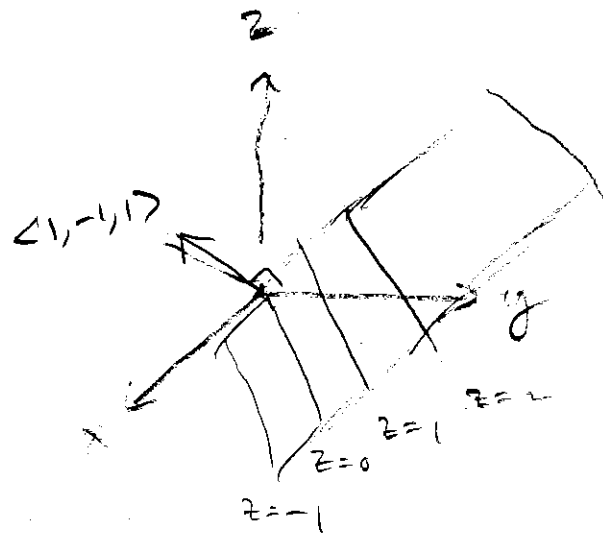
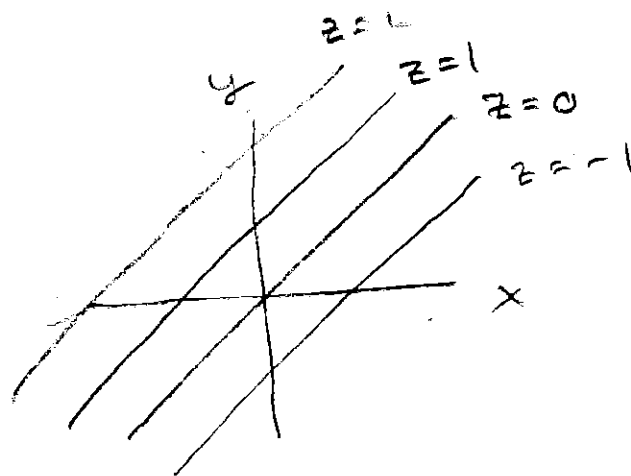
$$z=k$$

$$k = y - x \Rightarrow y = x + k$$

IT IS A PLANE !!

$$z = y - x \Leftrightarrow x - y + z = 0$$

$$\langle 1, -1, 1 \rangle = \vec{n}$$

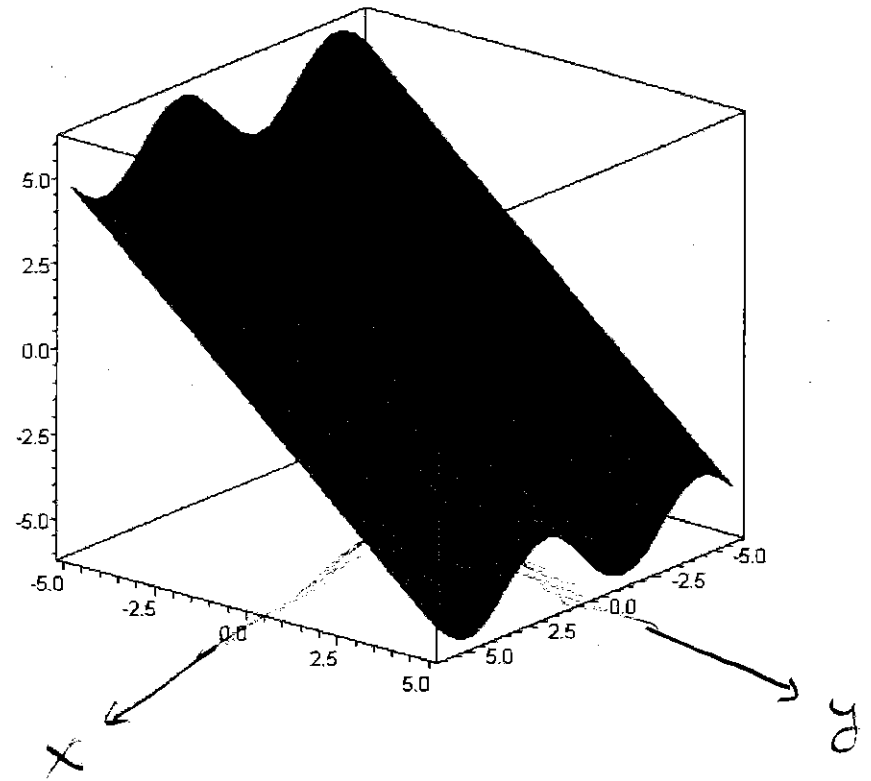
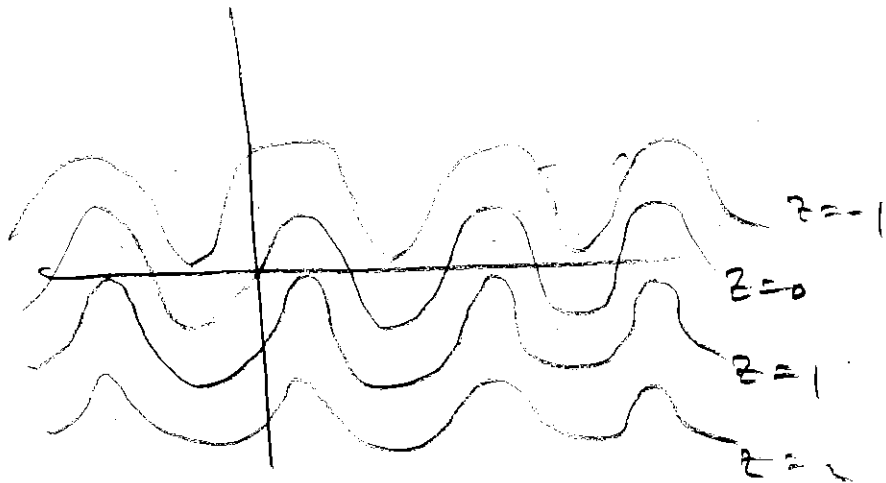


Example: Draw a contour map for  
 $z = \sin(x) - y$

$$z = k$$

$$k = \sin(x) - y$$

$$y = \sin(x) - k$$



Example: Draw a contour map for

$$z = f(x, y) = \frac{1}{1 + x^2 + y^2}$$

(use  $z = 1/10, 2/10, \dots, 9/10, 10/10$ )

$$k = \frac{1}{1 + x^2 + y^2}$$

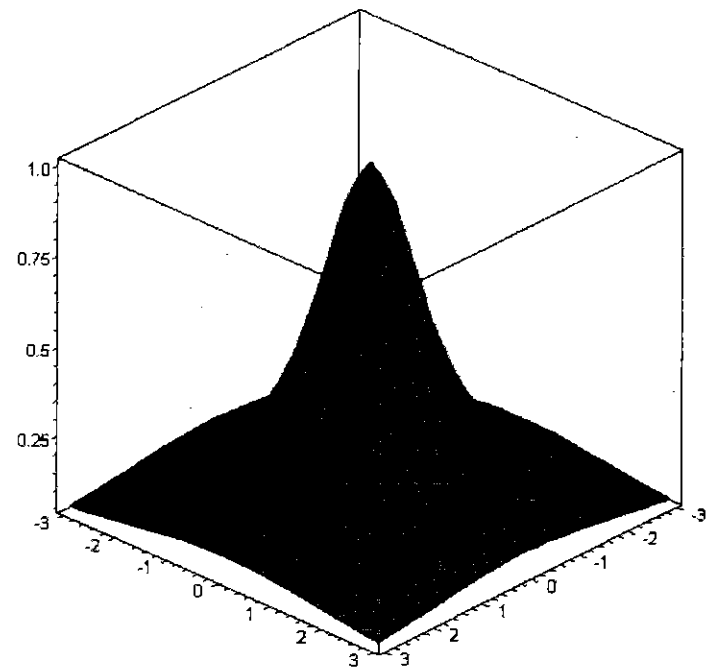
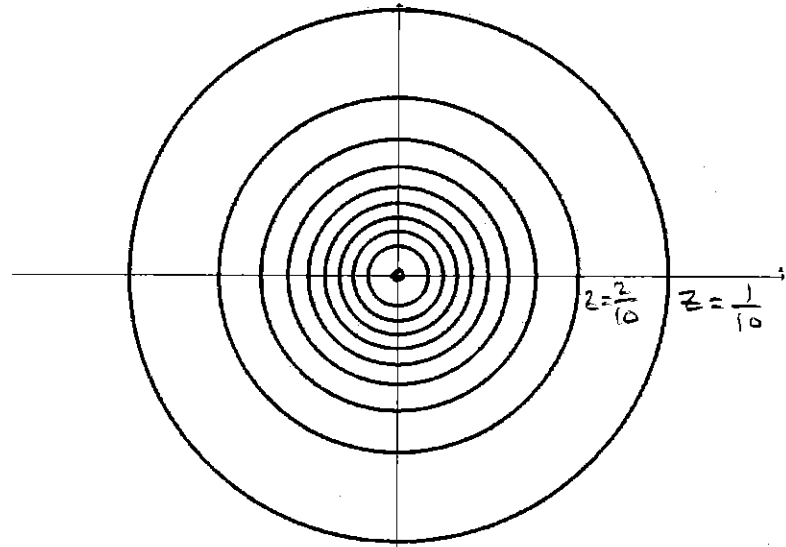
$$k(1 + x^2 + y^2) = 1$$

$$1 + x^2 + y^2 = \frac{1}{k}$$

$$x^2 + y^2 = \frac{1}{k} - 1$$

CIRCLE OF RADIUS  $\sqrt{\frac{1}{k} - 1}$

ONLY MAKES SENSE FOR  $0 < k \leq 1$



A question that asks “find the **domain**” is asking if you know your functions well enough to understand when they are defined and not defined.

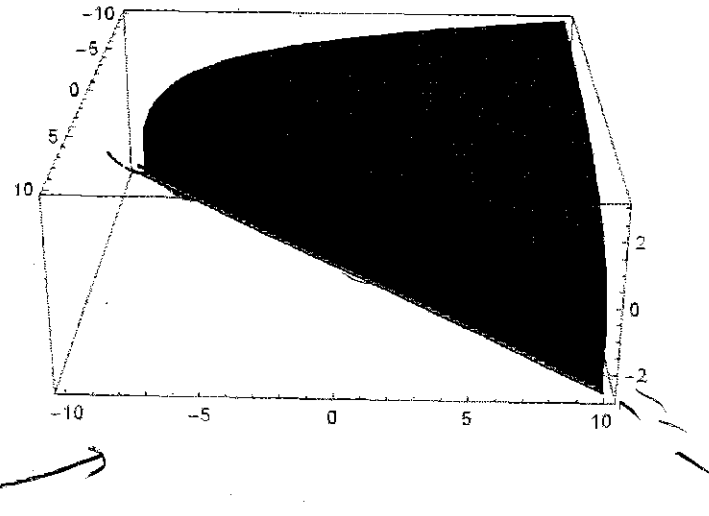
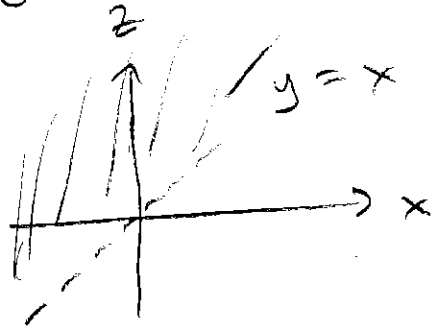
<i>Appears in Function</i>	<i>Restriction</i>
$\sqrt{BLAH}$	$BLAH \geq 0$
STUFF/BLAH	$BLAH \neq 0$
$\ln(BLAH)$	$BLAH > 0$
$\sin^{-1}(BLAH)$	$-1 \leq BLAH \leq 1$
and other trig...	

Examples: Sketch the domain of

(1)  $f(x, y) = \ln(y - x)$

$$y - x > 0$$

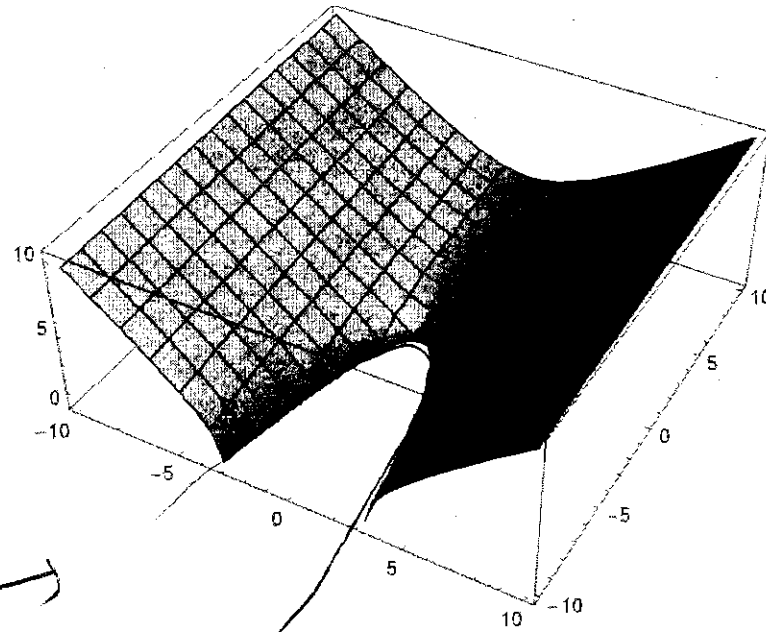
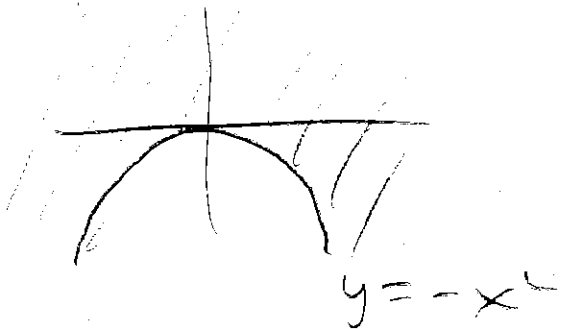
$$y > x$$



(2)  $g(x, y) = \sqrt{y + x^2}$

$$y + x^2 \geq 0$$

$$y \geq -x^2$$



## 14.3 Partial Derivatives

Goal: Get the slope in two different directions on a surface.

Recall the key def'n for all calculus

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Today we define:

$$\frac{\partial z}{\partial x} = f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$$

FIXED  
CONSTANT

$$\frac{\partial z}{\partial y} = f_y(x, y) = \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h}$$

FIXED  
CONSTANT

Motivation: Consider

$$f(x, y) = x^2y + 5x^3 + y^2$$

Find

$$\text{a. } \frac{d}{dx} [f(x, 2)] = \frac{d}{dx} [x^2(2) + 5x^3 + (2)^2]$$

$$\begin{array}{ccccccc} & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & 2 & 5 & 3 & 2 & 0 & 0 \\ = & 2 \times (2) & + & 5 \cdot 3x^2 & + & 0 & = 4x + 15x^2 \end{array}$$

$$\text{b. } \frac{d}{dx} [f(x, 3)] = \frac{d}{dx} [x^2(3) + 5x^3 + (3)^2]$$

$$\begin{array}{ccccccc} & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & 3 & 5 & 3 & 2 & 0 & 0 \\ = & 2 \times (3) & + & 5 \cdot 3x^2 & + & 0 & = 6x + 15x^2 \end{array}$$

$$\text{c. } \frac{d}{dx} [f(x, c)] = \frac{d}{dx} [x^2(c) + 5x^3 + (c)^2]$$

$$\begin{array}{ccccccc} & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & c & 5 & 3 & 2 & 0 & 0 \\ = & 2 \times (c) & + & 5 \cdot 3x^2 & + & 0 & = 2xc + 15x^2 \end{array}$$

$$\frac{\partial z}{\partial x} = 2xy + 15x^2$$

WHAT IS  $\frac{\partial z}{\partial y}$ ?

$$f_y(x, y) = x^2 \cdot 1 + 0 + 2y$$

$$\frac{\partial z}{\partial y} = x^2 + 2y$$



Example:

$$f(x, y) = x^3 y + \underbrace{x^5}_{\text{COEF.}} \underbrace{e^{xy^2}}_{\text{CONSTANT (NO X'S)}} + \ln(y)$$

$$\frac{\partial z}{\partial x} = 3x^2 y + \underbrace{5x^4}_{F'} \underbrace{e^{xy^2}}_S + \underbrace{x^5}_{F'} \underbrace{y^2 e^{xy^2}}_{S'} + 0$$

$$\frac{\partial z}{\partial x} = 3x^2 y + 5x^4 y^2 e^{xy^2}$$

$$f(x, y) = x^3 y + x^5 e^{xy^2} + \ln(y)$$

$$\frac{\partial z}{\partial y} = x^3 (1) + x^5 \cdot e^{xy^2} \cdot 2xy + \frac{1}{y}$$

$$\frac{\partial z}{\partial y} = x^3 + 2x^6 y e^{xy^2} + \frac{1}{y}$$

Example:

$$g(x, y) = \cos(x^3 + y^4)$$

$$g_x(x, y) = -\sin(x^3 + y^4) (3x^2 + 0)$$

$$g_x(x, y) = -3x^2 \sin(x^3 + y^4)$$

$$g_y(x, y) = -\sin(x^3 + y^4) \cdot (0 + 4y^3)$$

$$g_y(x, y) = -4y^3 \sin(x^3 + y^4)$$

## Important Note on Variables

A variable can be treated as:

1. A constant
2. An independent variable (input)
3. A dependent variable (output),

Examples:

a) **One variable function of x:**

$$\begin{array}{l} \text{OUTPUT} \curvearrowright \\ y = x^2 \\ \frac{dy}{dx} = 2x \\ \curvearrowleft \text{INPUT} \end{array}$$

b) **Related rates:**

At time  $t$  assume a particle is moving along the path  $y = x^2$

$$\begin{array}{l} x = x(t) \\ y = y(t) \\ \Leftrightarrow y(t) = (x(t))^2 \end{array}$$

$$\frac{dy}{dt} = 2x \frac{dx}{dt}$$

c) **Implicit functions:**  $x^2 + y^2 = 1$

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\begin{array}{l} y = y(x) \\ \Leftrightarrow x^2 + (y(x))^2 = 1 \\ 2x + 2(y(x)) \frac{dy}{dx} = 0 \\ \frac{dy}{dx} = \frac{-2x}{2y} \end{array}$$

d) Multivariable:  $z = x^2 + \underbrace{y^3 e^{6y}}_{\text{ALL CONSTANT}} - 5xy^4$  COEFF.

$z = x^2 + y^3 e^{6y} - 5xy^4$  CONSTANT COEFF.

$\frac{\partial z}{\partial x} = 2x + 0 - 5(1)y^4 = 2x - 5y^4$

↑ OUTPUT  
↑ INPUT

$\frac{\partial z}{\partial y} = 0 + \underbrace{y^3}_{F'} \cdot \underbrace{e^{6y}}_S \cdot 6 + \underbrace{3y^2}_{F'} \cdot \underbrace{e^{6y}}_S - 5x(4y^3) = 6y^3 e^{6y} + 3y^2 e^{6y} - 20xy^3$

e) Multivariable Implicit: CONSTANT

$x^2 + y^2 - z^2 = 1$   
 $x^2 + y^2 - (z(x)) = 1$

$\frac{\partial z}{\partial x} \cdot 2x + 0 - 2z \frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = \frac{-2x}{-2z} = \frac{x}{z}$

$\frac{\partial z}{\partial y} =$

$x^2 + y^2 - z^2 = 1$   
↑ ↑  
 CONSTANT  $z(y)$

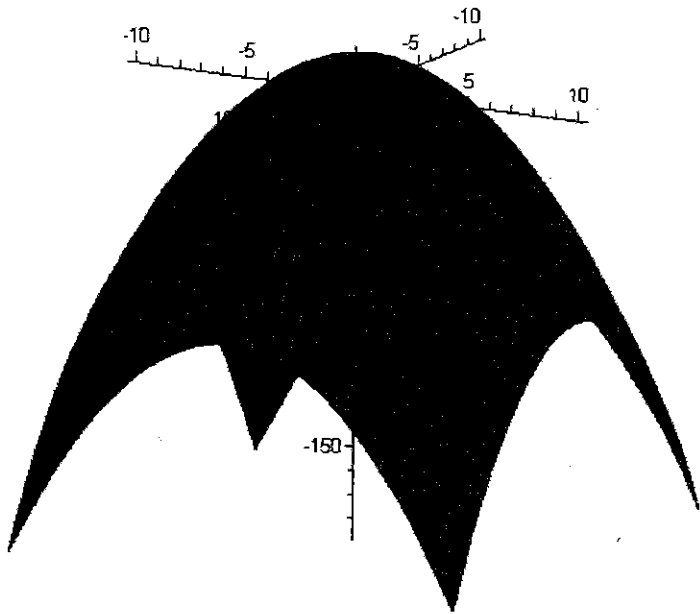
$0 + 2y - 2z \frac{\partial z}{\partial y} = 0$

$\frac{\partial z}{\partial y} = \frac{-2y}{-2z} = \frac{y}{z}$

## Graphical Interpretations:

Pretend you are skiing on the surface

$$z = f(x, y) = 15 - x^2 - y^2$$



$$f_x(x, y) = -2x$$

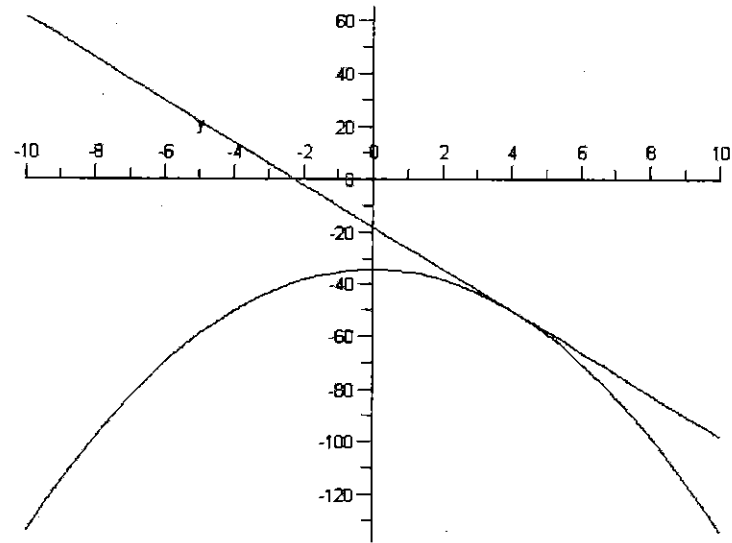
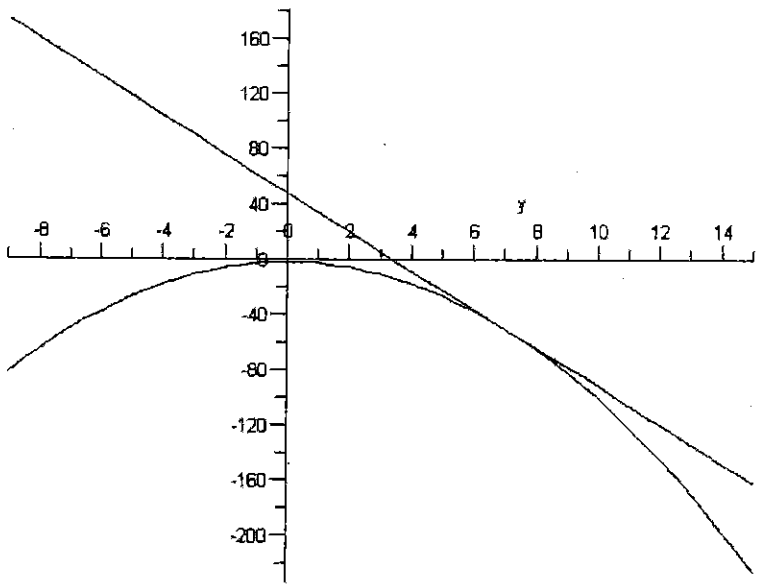
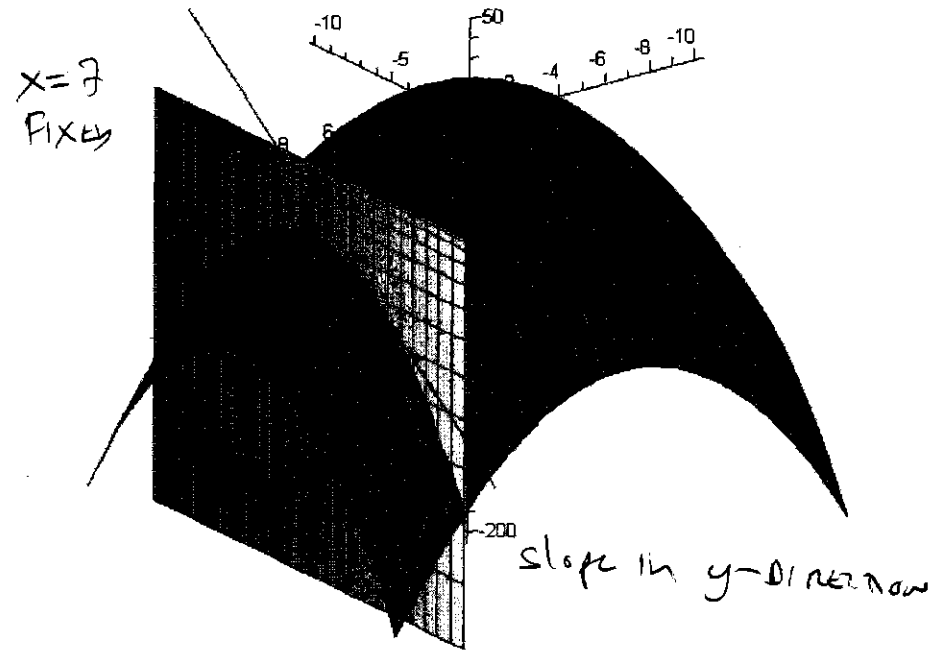
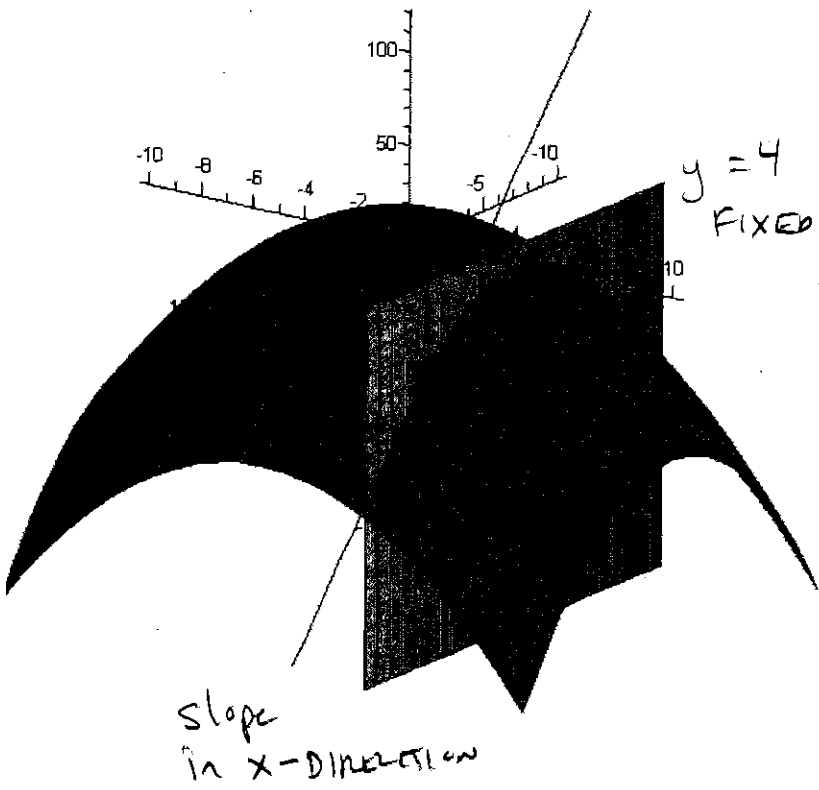
$$f_y(x, y) = -2y$$

$$f(4, 7) = 15 - (4)^2 - (7)^2 = 15 - 16 - 49$$

$$\boxed{f(4, 7) = -50} \quad \text{HEIGHT}$$

$$f_x(4, 7) = -8 \quad \text{"SLOPE IN } x\text{-DIRECTION"}$$

$$f_y(4, 7) = -14 \quad \text{"SLOPE IN } y\text{-DIRECTION"}$$



## Second Partial Derivatives

Concavity in x-direction:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right) = f_{xx}(x, y)$$

Concavity in y-direction:

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right) = f_{yy}(x, y)$$

Mixed Partial:

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) = f_{xy}(x, y)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) = f_{yx}(x, y)$$

Example: Find all second partials for

$$z = f(x, y) = x^4 + 3x^2y^3 + y^5$$

$$f_x = 4x^3 + 6xy^3$$

$$f_y = 9x^2y^2 + 5y^4$$

$$f_{xx} = 12x^2 + 6y^3$$

$$f_{xy} = 18xy^2 = f_{yx}$$

$$f_{yy} = 18x^2y + 20y^3$$



ALWAYS THE  
SAME!

(IF BOTH DEFINED AND CONTINUOUS)

CLAIRAUT'S THEOREM

(Written at 1<sup>st</sup> systematic discussion on  
3D calculus on surfaces)